DOCUMENT RESUME

ED 275 764 TM 860 681

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TITLE A Composite of Order Analysis Procedures.
PUB DATE Apr 85
NOTE 47p.; Paper presented at the Annual Meeting of the American Educational Research Association (66th, Chicago, IL March 31-April 4, 1985).
PUB TYPE Speeches/Conference Papers (150) -- Information Analyses (070) -- Reports - Research/Technical (143)

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS Analysis of Variance; Developmental Stages; Factor Analysis; *Learning Theories; *Literature Reviews; Matrices; *Multidimensional Scaling; Problem Solving; *Scaling; Sociometric Techniques; Test Items; Test Reliability
IDENTIFIERS Guttman Scales; Learning Hierarchies; *Order Analysis; *Ordering Theory

ABSTRACT

Order Analysis is a multidimensional scaling (MDS) technique for determining order among items. This paper reviews articles by different authors describing various components of ordering theory. A common nomenclature is constructed to link together the various ideas and is applied to a fairly simple set of data. Topics discussed include a more complete description of ordering theory in general and of order analysis in particular, a comparison of factor analysis with order analysis, an example of the use of order analysis in validating learning and instructional hierarchies, and a brief description of recent developments and concerns in ordering theory. Suggestions for applications of order analysis to the study of cognitive structures are included. (JAZ)

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A COMPOSITE OF ORDER ANALYSIS PROCEDURES

by

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April 1985

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Abstract

This paper reviews several articles written by different authors concerning the multidimensional scaling technique of order analysis. Various components of the theory, developed by different authors, are linked together by a common nomenclature. The order analysis procedure is then applied to a set of data. An examination of order analysis with respect to multidimensional scaling techniques is included as is a description of recent developments and concerns of ordering theory. Suggestions for application of order analysis to the study of cogniti[illegible]tructures are also included.

INTRODUCTION

From a cognitivist's view point, intellectual development is posited to take place in a sequential manner with mastery of a certain set of skills prerequisite to mastery of higher level skills. Piaget suggests that there are four sequential stages essential to the full development of an individual, and the mastery of the representative skills of a stage depend ultimately on the full and complete mastery of all skills in prerequisite stages. Bruner takes this one step further saying that an individual must go through sequential development of skills to attain mastery of not only general intellectual skills, but also to attain mastery of skills associated with any specific body of knowledge. Gagne' has used this idea to postulate that the material presented to the individual to be learned, must form an instructional hierarchy, with presentation and mastery of certain skills essential to mastery of skills higher in the developmental framework.

With this theory of intellectual development in mind, ordering theory was postulated as a means of discerning learning hierarchies. A measurement instrument may be viewed as a composite of a set of items, each tapping various skills important in attaining mastery of any body of knowledge. If one can discern the relationship among the items and the items truly represent the specified skills, it could be suggested that the same relationship holds among skills. Ordering theory was developed to assist in the essential step of finding order among

items.

This paper will attempt to provide a broad description of ordering theory and its recent developments. A common nomenclature to tie together the ideas developed by various authors is constructed and applied to a fairly simple set of data. Topics discussed include a more complete description of ordering theory in general and order analysis in particular, a comparison of factor analysis with order analysis, an example of the use of order analysis in validating learning and instructional hierarchies, and a brief description of recent developments and concerns in ordering theory. This paper looks closely at order analysis as a multi-dimensional scaling technique for defining order among items.

MULTIDIMENSIONAL SCALING

Multidimensional Scaling (MDS) is a technique developed in the behavioral and social sciences for studying the structure of objects or people. The basic data in MDS are measures of proximity, δ_{ij}, between pairs of objects i and j. A proximity measure is an index defined over pairs of objects which quantifies the degree to which the two objects are alike. If the highest values of δ_{ij} correspond to stimulus pairs which are most alike, then δ_{ij} is a measure of similarity. If the highest values correspond to stimulus pairs which are least alike, the δ_{ij} is a measure of dissimilarity. MDS refers to a set of multivariate statistical methods for estimating the parameters and assessing the fit of various spatial distance models for proximity data

(Davison, 1983).

Applications of MDS can be grouped into three categories, each marked by a different purpose. In dimensional applications, the researcher applies MDS to understand the stimulus dimensions to which the individual attends. The stimulus coordinates recovered in the scaling are interpreted as the locations of the objects along the salient aspects. In data reduction applications, the user's intent is to reduce the complex interrelationships between stimuli represented in the proximity matrix to a simpler, more readily comprehended, more readily communicated form. In the configural verification type of application, the user begins from an hypothesis which specifies the number of dimensions which should be obtained if the proximity measure is submitted to a nonmetric MDS, and it specifies the configuration which should be formed by the stimulus points.

Order Analysis is a multidimensional scaling technique (a point that has not often been made) utilizing dominance (i.e. how much more likely it is for an individual to answer item i correctly before answering item j correctly) rather than simply dissimilarity as a measure of proximity. Dominance implies an assymetric relationship between items. This paper investigates in some depth the two categories of applications dealing with definition of ordering dimensions and configural verification.

ORDERING THEORY

Ordering theory has as its primary intent either the testing of hypothesized hierarchies among items or the determination of hierarchies among items. The foremost example of the search for order among items is the work of Guttman. Guttman's Scalogram Analysis attempted to determine orderings among a set of test or questionnaire items.

> Scalogram Analysis is a deterministic model used to order a group of tasks into a linear hierarchy and to evaluate whether or not the hierarchy is unidimensional and cumulatively hierarchical. The degree to which a group of tasks is judged to be unidimensional and cumulative is determined by the extent to which 'passes' on any task co-occur with 'passes' on all tasks ranked lower in the network. Conversely, a hierarchical network is unidimensional and cumulative insofar as 'failures' on a task co-occur with 'failures' on all tasks ranked higher in the network. (Airasian & Bart, 1975)

The search was for an ordering in which items are arranged in a sequence such that a given response at one level in the sequence implied a specific set of responses at all lower steps of the sequence. Guttman's original technique has been elaborated and extended to multiple scaling techniques. Lingoes (1963) refers to a 'set theoretic model' for analyzing dichotomous items which he termed Multiple Scalogram Analysis. His algorithm includes the following steps:

1) Preparation of a dichotomous subject by item matrix (mxn)

2) Matrix reflection - The number of positive responses for each item are calculated for each of the n items. Those items for which column marginals are less than m/2 (less than one half of the subjects answered the item correctly) are reflected - 0's changed to 1's and 1's changed to 0's. Items which were answered correctly by everyone are eliminated.

3) Scale initialization - The item with the largest marginal sum is chosen as the initial item of a scale, by-passing items that have previously entered other scales. Only those items which are both significantly related to other items in the scale and are sufficiently close to an error free Guttman scale are considered. To test the fit of the original data to an error free Guttman scale, Lingoes defines a coefficient of reproducibility

$$REP = 1 - \{[\Sigma\Sigma|O_{ij} - G_{ij}|]/mn\}$$

where G is an error free Guttman matrix defined by never having more than two runs or sequences of 1's or 0's in any row, all rows with two runs having consistent ordering of 1's and 0's, and the data matrix O and the ideal matrix G being monotonically related.

Multiple Scalogram Analysis takes a sample of items and attempts to minimize the number of scales for a given set of relationships. It differs from scalogram analysis in that it is empirical, it may yield multiple scales when data demand it rather than rejecting the scale hypotheses for a set of items when it is treated as a whole and it has a statistical

rather than a heuristic decision basis for both grouping items and testing scale hypotheses. (Lingoes, 1963)

Table 1 displays the steps in the calculation of the coefficient of reproducibility for the item response matrix X, shown below.

Insert Table 1 About Here

The search for ordered scales previous to ordering theory had emphasized linear orderings. Because the larger the number of items in a linear ordering, the more difficult it is to obtain reproducibility.

> The most salient features of this scale (Guttman) are that the scale should be virtually homogeneous and that persons receiving the same summative score as an index of their location on a unidimensional continuum should have responded in the same way to all items. The objection that this criterion for scalability is rarely achieved even when total scores reach an acceptable level of reliability is partially met in ordering theory which demands the Guttman criterion of scalability be met only in parts of a test (in separate branches of ordering) and not in the test as a whole. (Bart & Krus, 1973)

With this and the fact that not all orderings among items must necessarily be of a linear nature, ordering theory was proposed. Ordering theory was suggested as a technique which permits investigation of both linear and non-linear ordering among items.

Ordering theory shares with classical models the item response matrix, but does not use summation across subject rows to express the subject's standing on the trait measured. Instead of using summative scores as a starting point for further statistical analysis, ordering theory is used to determine logical relationships between items represented in the item response matrix. An item response matrix, X, involving the response of 9 subjects to 5 items is shown in Table 2.

Insert Table 2 About Here

In studying logical relationships between items, the prerequisite relationship has been of primary concern in the development of ordering theory.

One requirement for the prerequisite relationship to be used to interrelate items is that the items be bi-valued - '1' is the item score for item i for subject k if subject k gives a correct response to item i and '0' is the item score for item i for subject k if subject k gives an incorrect response to item i. Success on item i is a prerequisite to success on item j if and only if the response pattern (01) for items i and j respectively does not occur. The response patterns (00), (10), and (11) are called confirmatory and the response pattern (01) is called disconfirmatory. (Bart & Krus, 1973) Besides the requirement that all tasks be dichotomously scored is that all subjects in a sample respond to all tasks.

ORDER ANALYSIS

Description of Procedures

Krus and Bart (1974) proposed an ordering theoretical method of multidimensional scaling of items. Order analysis, as they called the procedure, attempts to define the dimensions of the space containing the items of a given measurement instrument.

Item Dominance Matrix

Using the item response matrix for a set of n items, an n x n matrix may be constructed. This matrix is referred to as the item-dominance matrix by Krus (1977) and as the item-pattern matrix by Krus and Bart (1974). Letting X represent the item response matrix, the dominance matrix denoted by A may be defined by $A = X'\bar{X}$ where X' denotes the transpose of the item response matrix and $\bar{X}$ denotes the negation of the item response matrix (e.g. if $x_{ij}=0$, $\bar{x}_{ij}=1$; if $x_{ij}=1$, $\bar{x}_{ij}=0$).

Insert Table 3 About Here

Matrix A provides indices of the strengths of various inter-item prerequisite relationships. Cell entries, $a_{ij}=\sum_k x'_{ik}x_{kj}$ indicate the number of individuals that got item i correct but got item j wrong.

In multi-dimensional scaling terms, the elements of the dominance matrix may be thought of as dissimilarity measures. When a_{ij} is high, there is said to be less similarity between items i and j than when a_{ij} is low. The dissimilarity might be defined as the $\max(a_{ij},a_{ji})$ as the dimmilarity seems to be of a

directional nature, Baker and Hubert (1977) note this relationship by denoting the entries of the dominance matrix, a_{ij}, as measures of proximity (they use the notation $q(o_i,q_j)$).

Krus (1977) proposes a means of considering significantly non-random relationships in an order analysis and putting the items into a space where the distance between items is not a function of the number of subjects considered. In an attempt to sort 'real' relations from 'random' relations, Krus defined a statistic $z_{ij} = [a_{ij} - a_{ji}]/[a_{ij} + a_{ji}]^{1/2}$ (McNemar 1947 as reported in Krus 1977) which reflects the prerequisite relationship between items i and j. Assuming that z_{ij} is a standard normal random variable, he then determines the probability p_{ij} that the entries of the dominance matrix, a_{ij}, could have reached such a magnitude when in fact there is no prerequisite relationship between items i and j.

$$p_{ij} = \int_{-\infty}^{z_{ij}} (2\pi)^{-1/2} e^{-z_{ij}^2/2}\, dz_{ij}$$

The relation expressed by the cell entry a_{ij} is accepted if the corresponding p_{ij} value reaches or exceeds a prespecified level α. Krus then reduces the item dominance matrix A into a unit space by dividing 'significant' cell entries by the number of subjects m. He denotes this reduced matrix by $\tilde{A}$ with

$$\tilde{a}_{ij} = a_{ij}/m \text{ if } p_{ij} \geq \alpha$$
$$= 0 \quad \text{if } p_{ij} < \alpha$$

For example, if we set $\alpha = .70$ and examine the dominance matrix from Table 3, we arrive at the matrix displayed in Table 4.

Insert Table 4 About Here

A graph representing the relationship between items may then be constructed, a graphical display Krus refers to as a dendrogram.

> As in analytic geometry a geometric representation of algebraic structures can be constructed, in formal logic theory the graphic representations of logical relations can be drawn. The generalized manifest structure of attributes (items) is represented by a dendrogram, where an edge between nodes i and j is drawn if their corresponding element a_{ij} is nonzero. (Krus, 1977)

Baker and Hubert (1977) suggest that a directed graph be used to display the resultant item hierarchy. "The n items are to be considered as nodes in a two-dimensional plane. A directed 'edge' is drawn from o_i to o_j if and only if $q(o_i,o_j) \leq \epsilon$ - tolerance level set by the researcher (similar to a nonzero $\tilde{a}_{ij}$). The presence of both directed edges represents equivalence; with neither edge symbolizing independence."

Representing $\tilde{A}$ in a directed graph results in the following; a nonzero $\tilde{a}_{ij}$ indicating that item i is prerequisite to item j and the magnitude of $\tilde{a}_{ij}$ representing the hierarchy of the items. Figure 1 represents a directed graph of the data in Table 4.

Insert Figure 1 About Here

Subject Dominance Matrix

Items are now defined along a vertical dimension only. Krus (1977) proceeds to complete the solution by "isolating the latent structure of the data matrix attributes (item dominance matrix) where each node of the dendrogram is determined in its corresponding space". An attempt is made to locate each node (attribute or item) in a space defined by relevant factors.

> The factoring problem then relates to the determination of an independent set of linearly-ordered factors that will be determined by chains of two-item prerequisite relationships. (Krus & Bart, 1974)

In order to determine these factors or dimensions, we examine the ordering of subjects with regard to the number of items answered correctly. Using a procedure analgous to that used for determining the item dominance matrix A, a subject matrix E, may be constructed. The subject dominance matrix contains cell entries e_{ij} which equal the number of items to each of which subject i gave a correct response and subject j gave an incorrect response. Krus (1977) defines $E = X\overline{X}'$. Table 5 displays the subject dominance matrix for the data considered in Table 2.

Insert Table 5 About Here

Analgous to the item dominance matrix reduced to unit space., A, Krus defines a subject dominance matrix reduced to unit space E with

$$\tilde{e}_{ij} = e_{ij}/n \text{ if } p_{ij} \geq \alpha$$

$$= 0 \quad \text{if } p_{ij} < \alpha$$

where $z_{ij} = [e_{ij} - e_{ji}]/[e_{ij} + e_{ji}]^{1/2}$

$$p_{ij} = \int_{-\infty}^{z_{ij}} (2\pi)^{-1/2} e^{-z_{ij}^2/2} \, dz_{ij}$$

Tables 6 through 8 display these respective matrices.

Insert Tables 6 - 8 About Here

A non-zero entry in $\tilde{E}$ indicates a 'significant' ordering of subjects by the items in the measurement instrument. When a significant ordering of subjects is defined, we may use this ordering to define a dimension composed of items that successfully ordered these subjects.

The dominance matrices generalized into a unit space contain two basic types of information: first, information about the probability (greater than or equal to the prespecified level of α) of any of its elements to be followed by another element in a linearly ordered sequence, i.e. the probability a path will exist between an element e_{ij} and another element; and second, information about the distances between pairs of its constituent attributes (items) or entities (subjects). To isolate the longest possible linearly ordered sequences of the generalized dominance matrix elements, it was necessary to utilize both types of information simultaneously i.e., to search for elements with the highest probability of having a path to its closest neighbor.

By successive extraction of the largest possible groupings of

linearly ordered entities, the dendrogram structure was partitioned into its component branches. At the data matrix level, this operation corresponded to partitioning data into a set of k groups g, i.e., a set of assymetric, transitive, and 'disconnected' components. (Krus, 1977)

Figure 2 represents a directed graph of the data in Table 8.

Insert Figure 2 About Here

Item Loading Matrix

On the basis of these separate components, the original data matrix A was regrouped and partitioned into $X^{*\prime}=[X_{g1}',\ldots,X_{gk}']$. Every submatrix X_{gh} is then generalized into its latent space $A^*_{gh}=X'_{gh}\bar{X}_{gh}$ where $A = \sum_h A^*_{gh}$. For example, if E is reordered using the subject directed graph shown in Figure 2 the reordered matrix $\tilde{E}$ shown in Table 9 results.

Insert Table 9 About Here

Upon examination of the matrix $\tilde{E}$, the X matrix may be partitioned into the following two submatrices shown in Table 10.

Insert Table 10 About Here

In his discussion of problems of seriation using a subject by item response matrix, Hubert (1974) defines the conditions under

which a set of items constitutes a perfect Guttman scale.

> A perfect scale exists if and only if the rows of any matrix can be permuted so that the 1's in each column are consecutive.

By reordering the submatrices X_{g1} and X_{g2}, two perfect scales may be defined. Table 11 shows these two reordered matrices.

Insert Table 11 About Here

The item dominance matrices are then determined for the item response matrices defining the two perfect Guttman scales as shown in Table 12.

Insert Table 12 About Here

The two item-dominance matrices may be thought to reflect the dominance of the same set of items along two different dimensions. The row totals for each A^*_{gh} are then defined as the 'loading' of each item along the hth dimension, and may be combined into an item loading matrix C in which $cell_{ih}$ is an index of the association of item i to order dimension h. The resultant order loading matrix is shown in Table 13.

Insert Table 13 About Here

Matrix C can be viewed as analgous to a factor loading matrix, with column totals analgous to factor contributions

and row totals analgous to communalities. (Krus & Bart, 1974)

A two dimensional scaling representation for the order loading matrix is shown in Figure 3.

Insert Figure 3 About Here

Order Analysis As a Multidimensional Scaling Technique

After defining order dimensions and then scaling individual items along these dimensions using order analysis, an attempt was made to relate the findings to concepts contained within the multidimensional scaling (MDS) literature concerning nonmetric group solutions. Defining distance between items as $d_{ij}=\max|c_{ih}-c_{jh}|$ a dominance distance matrix D could be defined for the items A - E from our example. This matrix is shown in Table 14.

Insert Table 14 About Here

The calculated Euclidean distance $d_{ij}=(\sum_{h}(c_{ih}-c_{jh})^2)^{1/2}$ was about the same as the dominance distances as items lie fairly close to the axes of the two order dimensions except for items B and E.

An attempt was then made to compare this distance matrix to the initial dissimilarity matrix. However, order analysis has not clearly defined a matrix comparable to the dissimilarity matrix found in the MDS literature. The item dominance matrix A (the squared Euclidean distance between items) was first examined as a

possible candidate for a dissimilarity matrix. The fact that Hubert (1974) refers to this matrix as a similarity matrix, Baker and Hubert (1977) and Wise (1981) refer to it as a proximity matrix, and Krus (1977) refers to it as a dominance matrix reflects this confusion. The cell entries a_{ij}, in this matrix represent the number of subjects that provide the correct answer to item i but an incorrect answer to item j. If the rows and columns are arranged such that the row totals are in descending order and the column totals in ascending order, the entries in the upper triangular matrix might be interpreted as a measure of dissimilarity between items. For example, the rearranged matrix A is shown in Table 15.

Insert Table 15 About Here

The item E could be interpreted as more dissimilar from item D than item B because there are fewer individuals getting B incorrect when they pass item E than when they get item D incorrect.

As can be seen from this matrix, it is assymetric, reflecting possibly directional distance between items (dominance). If the upper triangular matrix of the corresponding dominance distance matrix D is compared to the matrix A, as in Table 16, a monotonic relationship seems to exist between the distance measure and the cell entry for the item dominance matrix for any given column, i.e. $a_{ij}<a_{i'j}$ $d_{ij}<d_{i'j}$ for all j, except for a_{AD}.

Insert Table 16 About Here

However, this does not hold between columns. For example $a_{EC} = 4 < a_{ED} = 5$ and $d_{EC} = 6 = d_{ED} = 6$. Therefore, distance is not strictly monotonically related to item dominance.

If there is to be any measure of fit comparable to those defined in MDS e.g. Stress = $\sqrt{[\sum\sum(\hat{\delta}_{ij} - \hat{d}_{ij})^2]/\sqrt{\sum\sum\hat{d}_{ij}^2}}$ or Coefficient of Monotonicity $\mu = [\sum\sum\hat{\delta}_{ij}\hat{d}_{ij}/\sqrt{(\sum\sum\hat{\delta}_{ij}^2)(\sum\sum\hat{d}_{ij}^2)}$, where $\hat{\delta}_{ij}$'s are disparities which mirror the rank order of the data and the $\hat{d}_{ij}$'s representing estimates of distance between items (Davison, 1983), a disparity matrix with cell entries $\hat{\delta}_{ij} = f(\hat{d}_{ij})$ which meets the demands of monotonicity for a suitably defined dissimilarity (dominance) matrix ($\delta_{ij} < \delta_{i'j'}$, $\hat{\delta}_{ij} < \hat{\delta}_{i'j'}$) will have to be defined. Candidates for consideration as dissimilarity matrices might include $\tilde{A}$ or probability matrix P_A. Until that time, many of the procedures such as tests of goodness of fit used in MDS for defining dimension will not be directly applicable.

Comparison of Order Analysis with Factor Analysis

Several of the articles reviewed proceed to make a comparison between the factors resulting from a factor analysis and the dimensions resulting from an order analysis. Bart (1978) performed an empirical inquiry into the relationship between test factor structure and test hierarchical structure. He concluded that sets of items with good, simple, clear factor structure did

not represent clear simple hierarchical structure.

Krus and Weiss (1976) also looked into this relationship and found that the degree of congruence for factor and order analytic solutions appears to be jointly determined by the α parameter of order analysis (e.g. α =.95 analyzes only orders which are nonrandom with the degree of certainty equal to or greater than .95).

> If the data structures were highly organized (nonrandom), order analysis at any α level and factor analysis would frequently converge. If the data matrix contained substantial amounts of random variation, only the order analytic solution at a low α level converged with the simultaneous factor analytic solution. An order analytic solution at a higher level typically indicated absence of any practical consequences. (Krus & Weiss, 1976)

They go on to conclude, "The most important property of order analysis appears to be its insensitivity to random variation patterns as compared with factor analytic models".

Krus (1978) proposes the reason for the difference between factor analysis and order analysis is that the dimensions derived by factor analysis are based on both proximity and dominance relations at the item level and only on proximity relations at the factor extraction level. The dimensions extracted by order analysis are based solely on dominance relations at both the item and factor extraction levels.

For the binary case, let the frequencies of response patterns between two variables, A and Y, be written as a, b, c, and d,

where a, b, c, and d stand for a frequency of the (11), (00), (10), and (01) response patterns, respectively, a and b response patterns (11) and (00) can be labeled as a proximity type, and c and d (10) and (01) as a dominance type.

Factor analysis typically analyzes structures based on matrices of coefficients of correlation. Yule's formula for the product-moment coefficient of correlation in a binary form can be written as

$$\phi = [ab - cd]/[(a+d)(b+c)(c+a)(d+b)]^{1/2}.$$

The resulting factorial dimension is interpreted in terms of intercluster proximities of the analyzed variables, independently of whether these proximities were generated by relations of proximity, dominance, or both.

Order analysis was designed primarily for analysis of matrices of dominance matrices. The estimation of the dominance coefficients can be written as

$$\delta = [c - d]/[c + d]^{1/2}.$$

In order analysis, a direction hypothesis is formed, and the data matrix is analyzed with respect to dominance relations. Thus, order analysis can be either similar or dissimilar to factor analysis depending on the character of the data analyzed. (Krus, 1978)

The greater the degree that responses are of a dominance nature, the closes the results of factor analysis and order analysis will be.

Uses of Order Analysis

Order analysis is used primarily as an exploratory scaling technique in which an attempt is made to determine the number of dimensions needed to represent hierarchies among items. A confirmatory scaling technique is designed to test an a priori hypothesis about the ordering of psychological variables along one or more dimensions. Once hierarchies among items are arrived at empirically through order analysis, comparison may then be made to an a priori hypothesis of ordering among items.

Bart and Airasian (1975) suggest that there are three aspects of the two hierarchies which can be compared for validation purposes:

1) the probability that the total number of prerequisite relationships observed in a hierarchy could have occurred by chance
2) comparison of the specific prerequisite relationships posited a priori with those observed from an analysis which is performed independent of the pre-specified relationships, and
3) whether those tasks hypothesized to be independent of each other are in fact, independent.

The probability that the total number of prerequisite relationships observed could have occurred by chance is a function of the proportion of subjects who successfully complete the task, the number of tasks, the tolerance of the analysis (ϵ), and the number of subjects.

When many tasks are at the extreme end of the difficulty

> continuum, that is, when either very many or very few subjects pass the tasks, it is likely that many prerequisite relationships will be manifested, more because of the extremeness of the task difficulties than because of the existence of the prerequisite relationships. Similarly, the number of tasks investigated and the tolerance level for accepting inter-task relationships have a direct bearing upon the total number of prerequisite relationships observed. Finally, different sample sizes can have a differential effect upon the probability of observing a prerequisite relationship between two tasks. (Airasian & Bart, 1975)

Wood (as reported in Airasian and Bart (1975)) defined and tested an algorithm which estimates the probability distribution for the number of prerequisite relationships observed between tasks in a hierarchy, taking into account these four parameters.

> The Wood test which uses a Poisson distribution to approximate a mixture of binomial distribution Goodness of Fit tests, comparing the probability distribution obtained from the algorithm with probability distributions generated from computer simulations, indicates good fit for tolerance levels below 5 percent.

The Wood test serves as a useful screening technique in the validation process, as it provides an estimate of the chance likelihood of observing the total number of prerequisite relations actually observed.

Airasian and Bart suggest that when testing whether an a priori hierarchy fits a set of data, all possible prerequisite

relationships should be considered.

> Having hypothesized a learning hierarchy, constructed a performance test for the tasks in the hierarchy, and administered the test to students, the researcher should approach his data as though no hypothesized, pre-defined hierarchy existed. He should try to reconstruct the best-fitting hierarchy given the observed data, but independent of the hypothesized hierarchy. (Airasian & Bart, 1975)

When the reconstruction is completed, the researcher should refer to the initial hypothesized hierarchy and check the correspondence between the hypothesized and empirical hierarchies.

> In addition to simply checking to confirm the existence of hypothesized prerequisite relations, he should determine whether tasks hypothesized to be independent of one another are, in fact, verified as being independent. It is only after consideration of the task difficulty levels and the prerequisite and independence relations existing among <u>all</u> tasks, that an a priori learning hierarchy can be claimed to be valid. (Airasian & Bart, 1975)

Davison (1980) proposes such a confirmatory scaling technique by suggesting a method that first tests for quasi-independence between items. If the items are indeed related, a χ^2 goodness of fit test is performed to test whether the relationship between items can be accounted for by the proposed ordering.

For an a priori order hypothesis to be testable via these

methods, an order hypothesis must divide the set of potential subject response patterns into two non-overlapping, complimentary sets: a set of admissable response patterns and a set of inadmissable response patterns. (A subject's response pattern is a vector containing a subject's scores on response variables whose ordering is specified in the hypothesis.) The set of admissable response patterns is the set of score vectors which satisfy the order hypothesis. (Davison, 1980)

Letting S represent inadmissable patterns, R admissable response patterns, T response patterns limited by task demands, and $\pi(x) = \pi(a,b,c,\ldots)$ being the probability that a subject randomly selected from the population of interest has a score of a on variable A, b on B,...,

$\pi(x) = 0 \qquad \forall x \in T$

$\pi(x) = q(a) \quad q(b) \quad \ldots \forall x \in S$ ($q(\cdot)$ are usually defined as marginal probabilities)

$\pi(x) = \delta q(a) \quad \ldots \qquad \forall x \in R$

The model is fit to the frequencies through use of loglinear estimates of δ, $q(a)$,...

The order hypothesis would be fully supported only if the quasi-independence model were rejected, the scaling model could not be rejected, and the estimate of $\delta > 1.00$. Such a combination of findings would suggest that there were additional dependencies among the response variables beyond those imposed by the task, and that the order hypothesis

could account for those additional dependencies. (Davison, 1980)

Airasian and Bart point to ordering theory as a means for validating a priori instructional hierarchies.

> Crucial to the investigation of instructional hierarchies is the need to demonstrate that the hypothesized prerequisite relations among tasks in a hierarchy are confirmed by student learning data. (Airasian & Bart, 1975)

Airasian and Bart compared the a priori learning hiearchy proposed by Gagne',et al. for performing two tasks dealing with addition of integers. The hierarchy resulting for order analysis was some degree different from that proposed by Gagne'.

Bart and Mertens (1979) investigated the hierarchical structure of Piaget's formal operations.

> The hierarchical structure of the formal operations period of Piaget's theory of cognitive development was explored through the application of ordering theoretic methods to a set of data that systematically tapped the various formal operational schemes. The results suggest that the tasks within some schemes are empirically equivalent. While the response patterns are quite varied, the results do suggest that some common structure may underlie performance on the tasks, thus supporting Piaget's notion of the integrative structure of the period. (Bart & Mertens, 1979)

FURTHER DEVELOPMENTS IN ORDERING THEORY

This last section of the paper consists of a brief summary of some of the developments and present concerns of ordering theory as reflected in recent articles: hierarchical scaling of variance space, the question of consistency in ordering, and the comparison of proximity matrices for different populations.

Variance

Krus and Ceuvorst (1978) discuss a procedure for defining a "hierarchical scaling of variance". This procedure involves partitioning the total dominance among scores (analgous to sum of squares) into that due to dominance among individuals and that due to dominance among items. Let x_{ij} equal the score of subject i on item j (it does not necessarily have to be of a dichotomous nature). Table 17 gives an item response matrix for 5 subjects on 5 items.

Insert Table 17 About Here

If in general we have m subjects and k items, we first break down the dominance due to subjects by constructing an mxm matrix P such that if $p_{ii'} = \sum_{j}(x_{ij}-x_{i'j})$ such that if $x_{ij} < x_{i'j}$, then $(x_{ij} - x_{i'j}) = 0$. Table 18 gives the P matrix for the item response matrix in Table 17. (e.g. $p_{14}=(9-3)+(7-4)+(6-5)+0+0=10$)

Insert Table 18 About Here

A net dominance matrix Q is then constructed such that

$q_{ii'}=p_{ii'}-p_{i'i}$ if $p_{ii'} > p_{i'i}$ and 0 otherwise.

Insert Table 19 About Here

They then define a matrix $S_{subject}$ such that $S_{subject} = (q_{ii'}^2)$.

Insert Table 20 About Here

Dominance among subjects is given by $\sum\sum_{i\,i'} s_{ii'}=1214$. This then is used to calculate an unbiased estimate of variance due to subject $\hat{\sigma}^2_{sub} = [\sum\sum_{i\,i'} s_{ii'}]/[mn\,(n-1)] = 12.14$.

Similarly, a net dominance matrix H among items may also be constructed, along with an $S_{item} = (h_{jj'}^2)$.

Insert Table 21 About Here

Dominance among items is given by $\sum\sum_{j\,j'} s_{jj'} = 74$ with $\hat{\sigma}^2_{item}=[\sum\sum_{j\,j'} s_{jj'}]/[mn\,(m-1)] = .74$.

Dominance among scores in general may be determined by constructing an nxn $(k=mn)$ matrix D where $d_{ij} = x_i - x_j$ if $i>j$ and 0 otherwise; $S_{total}=(d_{ij}^2)$. Total dominance is given by $\sum\sum s_{ij}=2944$. The maximum likelihood estimate of total variance is given by $\hat{\sigma}^2_x = [\sum\sum_{i\,j} s_{ij}]/[k^2] = 4.7104$.

From this, we may then form an analysis of variance table with residual dominance determined by subtraction.

Insert Table 22 About Here

(The analysis seems to indicate that subject and item dominance are not significant due to the large residual dominance.)

Consistency

Bart (1974) suggests needed modification to the traditional concepts of validity and reliability. He proposes developing concepts of structural stability, structural equivalence, and structural validity.

> A test has structural stability to the extent to which it will engender the same item ordering at time T_2 that it did at time T_1 with the same subjects being used in both cases. A possible coefficient of structural stability would be the percentage of item response patterns obtained at time T_2 that are compatable with the ordering generated for the items at time T_1.
>
> A test has structural equivalence to the extent to which the ordering for a parallel form of a test generates the same ordering as the initial test given that a common group of subjects is used in the testing.
>
> A test has structural validity to the extent to which the test has an item ordering which is either a hypothetical theoretical ordering or an ordering possessed by some criterion test of items (analgous to concurrent validity). (Bart, 1974)

Cliff (1977) developed two sets of indices of consistency; one for items and one for subjects. Consistency of items will be considered here. Using the dominance matrix for items, A, calculated from the item response matrix, he defines a fundamental measure of dominance for items $\nu = \sum\sum_{jj'} a_{jj'}$. He then constructs a net dominance matrix similar in procedure to that used by Krus and Ceurvorst in calculating their matrix Q. Letting the entries in this matrix be denoted as $q_{jj'}$, $\nu_m = \sum\sum_{jj'} q_{jj'}$. ν_m is shown to be the lower limit for ν when items with these marginals are perfectly consistent. Table 23 displays the net dominance matrix for the item response matrix of Table 17.

Insert Table 23 About Here

If all items are independent, the expected value of ν is shown to be $\nu_c = 1/n[\sum_j x_{.j} \sum_j \bar{x}_{.j} - \sum_j x_{.j}\bar{x}_{.j}]$ where n is the number of items, and $x_{.j}$ (number of individuals getting item j wrong) and $\bar{x}_{.j}$ (number of individuals getting item j right) are the common marginals for the X and $\bar{X}$ matrices defined earlier. The measure of consistency is then given as $c_{t3} = [\nu_c - \nu]/[\nu_c - \nu_m]$ which is noted to be identical to Loevinger's homogeneity index. The greater the consistency index varies from 1, the less the degree of consistency of items for a particular set of subjects. For the response matrix of Table 17, $\nu = 36$, $\nu_m = 67$, $\nu_c = 84.4$, and $c_{t3} = 2.78$.

Similarity of Proximity Matrices

Schultz and Hubert (1976) proposed a nonparametric test of the hypothesis that two proximity matrices on the same set of n variables or objects reflected a similar patterning of high and low entries. "Proximity is assumed to be a positive symmetric measure for which the larger numerical values are assigned to the more dissimilar object pairs," (Schultz & Hubert, 1976). Baker and Hubert (1977) refered to the item dominance matrix A as a proximity matrix.

The null hypothesis H_o to be tested suggests that there is a random correspondence between two proximity matrices. As an example of this test, the two proximity matrices shown in Table 24, with cell entries $a_{ij}^{(1)}$ for item dominance matrix A^1 and $a_{ij}^{(2)}$ for item dominance matrix A^2, will be examined.

Insert Table 24 About Here

The index Γ measures the amount of common patterning of high and low entries across A^1 and A^2, and in particular, will be relatively large if the higher entries from $A^1$and A^2 tend to be in the same matrix position.

$$\Gamma = \sum_i \sum_j a_{ij}^{(1)} a_{ij}^{(2)}$$

Γ may be interpreted as an unnormalized correlation coefficient between the elements of A^1 and A^2. In our example $\Gamma = 31$.

A test statistic Z is defined using product moments of the distribution derived by Mantel (1967).

$$E(\Gamma) = [1/(n(n-1))]VP_1Q_1$$

$$Var(\Gamma) = -[1/(n(n-1))]^2P_1Q_1 + [2/(n(n-1))]P_3Q_3 +$$
$$[4/(n(n-1)(n-2))][P_2-P_3][Q_2-Q_3] +$$
$$[1/(n(n-1)(n-2)(n-3)][P_1-4P_2+2P_3][Q_1-4Q_2+2Q_3]$$

where

$$P_1 = (\Sigma\Sigma a_{ij}^{(1)})^2 \qquad Q_1 = (\Sigma\Sigma a_{ij}^{(2)})^2$$

$$P_2 = \Sigma(\Sigma a_{ij}^{(1)})^2 \qquad Q_2 = \Sigma(\Sigma a_{ij}^{(2)})^2$$

$$P_3 = \Sigma\Sigma(a_{ij}^{(1)})^2 \qquad Q_3 = \Sigma\Sigma(a_{ij}^{(2)})^2$$

It is assumed that the variables in A^1 are fixed but that all n! permutations of the rows (and simultaneously, columns) of A^2 are equally likely under the assumption of no comparable patterning common to A^1 and A^2.

In our example:

$P_1 = 1296$ $\quad Q_1 = 961$

$P_2 = 356$ $\quad Q_2 = 211$

$P_3 = 118$ $\quad Q_3 = 73$

$E(\Gamma) = 55.8$

$Var(\Gamma) = 174.06$

The normalized statistic $Z = (\Gamma - E(\Gamma))/[var(\Gamma)]^{1/2}$ can be compared conservatively to the values generated from the traditional Chebychev or Cantelli bounds. The Cantelli inequality states that the probability of observing a value larger than Z is $1/(Z^2+1)$. In our example: $Z = -1.8798$, $1/(Z^2+1) = .2206$. Based upon these results, the hypothesis that there is a random correspondence between the two proximity matrices can not be rejected.

DISCUSSION

Order analysis is a valuable multidimensional scaling (MDS) technique for determining order among items. However, for it to be fully functional, it should be viewed as a multidimensional scaling technique with appropriate reference to various concepts in the MDS literature. Only through an attempt to link the two bodies of knowledge can order analysis make use of already existing methods of defining dimension appropriate for order analysis. This paper suggests possible missing links between the two sets of literature.

In addition, this paper provides a necessary integration of the various ordering techniques through a common nomenclature. When ordering theory techniques become complete enough that they can be used confidently in establishing learning and instructional hierarchies, a wealth of opportunities to improve instruction will be available. If an instructional hierarchy could be established and a measurement instrument used to locate an individual at a level in the hierarchy, instruction could be prescribed to assist the individual in acquiring the prerequisite skills for reaching higher into the network and eventually attaining mastery. Ordering theory seems to offer a valuable diagnostic tool for providing individuals with adaptive instruction.

Much research in problem solving in mathematics has concerned itself with determining the differences between 'novice' and 'expert' problem solvers (Heller & Greeno, 1980; Krutetskii, 1976). A glimpse of the cognitive framework of different groups

might be possible through construction of a learning hierarchy of various problem-solving skills. Through study of the hierarchies of problem-solving process, insight into the acquisition of the knowledge base essential for adequate solution of a problem may be attainable. As well as proficiency in problem solving, examination of problem solving skill of groups differing in level of math anxiety might be studied (Druva, 1984). It might be suggested that there exists a variation in cognitive schema between math-anxious and non-math-anxious individuals - in the various content components of the cognitive framework as well as the linkages connecting these components. Application of a general framework or schema to understand novel problem-solving situations might provide the math-anxious individual with the required sense of being in control and thus avoid heightened debilitating anxiety.

Table 1
Calculation of the Coefficient of Reproducibility

Item Response Matrix, X

		Items				
		A	B	C	D	E
Subjects	1	1	1	0	1	1
	2	1	0	0	0	1
	3	0	1	1	0	1
	4	1	1	1	1	1
	5	1	1	1	0	1
	6	1	0	0	0	0
	7	0	1	0	0	1
	8	0	0	0	0	0
	9	1	1	0	0	1
Column Total		6	6	3	2	7

Reflected Matrix

1	1	1	0	1
1	0	1	1	1
0	1	0	1	1
1	1	0	0	1
1	1	0	1	1
1	0	1	1	1
0	1	1	1	1
0	0	1	1	0
1	1	1	1	1

Reordered Original Data Matrix, O

	D	C	B	A	E
4	1	1	1	1	1
5	0	1	1	1	1
1	1	0	1	1	1
9	0	0	1	1	1
3	0	1	1	0	1
7	0	0	1	0	1
2	0	0	0	1	1
6	0	0	0	1	0
8	0	0	0	0	0

Guttman Matrix, G

D	C	B	A	E
1	1	1	1	1
0	1	1	1	1
0	1	1	1	1
0	0	1	1	1
0	0	1	1	1
0	0	0	1	1
0	0	0	1	1
0	0	0	0	1
0	0	0	0	0

REP = 1 - 8/45 = .82

Table 2
Item Response Matrix, X

		Items				
		A	B	C	D	E
Subjects	1	1	1	0	1	1
	2	1	0	0	0	1
	3	0	1	1	0	1
	4	1	1	1	1	1
	5	1	1	1	0	1
	6	1	0	0	0	0
	7	0	1	0	0	1
	8	0	0	0	0	0
	9	1	1	0	0	1

Table 3
Calculation of Item Dominance Matrix, A

$$X' = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad \bar{X} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$X'\bar{X} = A = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 2 & 4 & 4 & 1 \\ 2 & 0 & 3 & 4 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 4 & 5 & 0 \end{bmatrix} \end{array}$$

Table 4
Calculation of Reduced Item Dominance Matrix, $\tilde{A}$

$$Z = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1.3 & 2.0 & -0.6 \\ 0 & 0 & 1.7 & 2.0 & -1.0 \\ -1.3 & -1.7 & 0 & 0.6 & -2.0 \\ -2.0 & -2.0 & -.6 & 0 & -2.2 \\ 0.6 & 1.0 & 2.0 & 2.2 & 0 \end{bmatrix} \end{array} \qquad P_A = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} .50 & .50 & .91 & .98 & .28 \\ .50 & .50 & .96 & .98 & .16 \\ .09 & .04 & .50 & .72 & .02 \\ .02 & .84 & .98 & .99 & .50 \\ .72 & .84 & .98 & .99 & .50 \end{bmatrix} \end{array}$$

$$\tilde{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & .44 & .44 & 0 \\ 0 & 0 & .33 & .44 & 0 \\ 0 & 0 & 0 & .22 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ .22 & .11 & .44 & .56 & 0 \end{bmatrix} \end{array}$$

Figure 1
Directed Graph of Items A - E

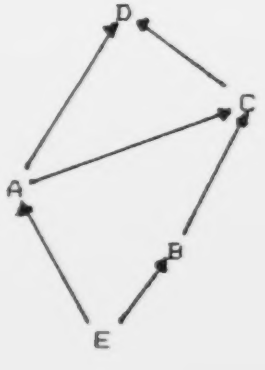

Table 5
Construction of Subject Dominance Matrix, E

$$X = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad X' = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$E = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 0 & 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 & 0 & 3 & 1 & 3 & 1 \\ 1 & 3 & 2 & 0 & 1 & 4 & 3 & 5 & 2 \\ 1 & 2 & 1 & 0 & 0 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 & 1 & 3 & 0 \end{bmatrix} \end{array}$$

Table 6
Standard Normal Matrix, Z

$$Z = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 0 & 1.41 & 0.58 & -1.00 & 0 & 1.73 & 1.41 & 2.00 & 1.00 \\ -1.41 & 0 & -0.58 & -1.73 & -1.41 & 1.00 & 0 & 1.41 & -1.00 \\ -0.58 & 0.58 & 0 & -1.41 & -1.00 & 1.00 & 1.00 & 1.73 & 0 \\ 1.00 & 1.73 & 1.41 & 0 & 1.00 & 2.00 & 1.73 & 2.24 & 1.41 \\ 0 & 1.41 & 1.00 & -1.00 & 0 & 1.73 & 1.41 & 2.00 & 1.00 \\ -1.73 & -1.00 & -1.00 & -2.00 & -1.73 & 0 & -0.58 & 1.00 & -1.41 \\ -1.41 & 0 & -1.00 & -1.73 & -1.41 & 0.58 & 0 & 1.41 & -1.00 \\ -2.00 & -1.41 & -1.73 & -2.24 & -2.00 & -1.00 & -1.41 & 0 & -1.73 \\ -1.00 & 1.00 & 0 & -1.41 & -1.00 & 1.41 & 1.00 & 1.73 & 0 \end{bmatrix} \end{array}$$

Table 7
Subject Probability Matrix, P_E

$$P_E = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} .50 & .92 & .72 & .34 & .50 & .96 & .92 & .98 & .84 \\ .42 & .50 & .22 & .46 & .42 & .84 & .50 & .92 & .34 \\ .22 & .72 & .50 & .42 & .34 & .84 & .84 & .96 & .50 \\ .84 & .96 & .92 & .50 & .84 & .98 & .96 & .99 & .92 \\ .50 & .92 & .84 & .34 & .50 & .96 & .92 & .98 & .84 \\ .46 & .34 & .34 & .48 & .46 & .50 & .22 & .84 & .42 \\ .42 & .50 & .34 & .46 & .42 & .72 & .50 & .92 & .34 \\ .48 & .42 & .46 & .49 & .48 & .34 & .42 & .50 & .46 \\ .34 & .84 & .50 & .42 & .34 & .72 & .84 & .96 & .50 \end{bmatrix} \end{array}$$

Table 8
Reduced Subject Dominance Matrix, $\tilde{E}$

	1	2	3	4	5	6	7	8	9
1	0	.40	.40	0	0	.60	.40	.80	.20
2	0	0	0	0	0	.20	0	.40	0
3	0	.40	0	0	0	.60	.20	.60	0
4	.20	.60	.40	0	.20	.80	.60	1.00	.40
5	0	.40	.20	0	0	.60	.40	.80	.20
6	0	0	0	0	0	0	0	.20	0
7	0	0	0	0	0	.40	0	.40	0
8	0	0	0	0	0	0	0	0	0
9	0	.20	0	0	0	.40	.20	.60	0

Figure 2
Subject Directed Graph

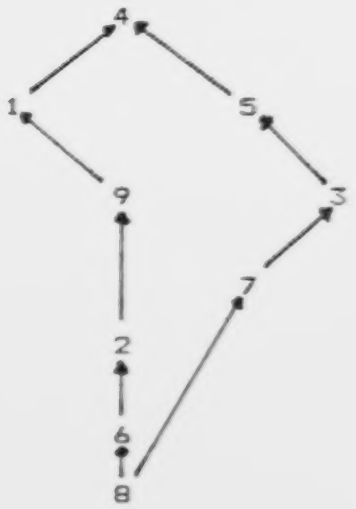

Table 9
Reordered Subject Dominance Matrix, $\tilde{E}$

$$\tilde{E} = \begin{array}{c} \\ 4 \\ 1 \\ 9 \\ 2 \\ 6 \\ 8 \\ 5 \\ 3 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 4 & 1 & 9 & 2 & 6 & 8 & 5 & 3 & 7 \end{array} \\ \begin{bmatrix} 0 & .20 & .40 & .60 & .80 & 1.00 & .20 & .40 & .60 \\ 0 & 0 & .20 & .40 & .60 & .80 & 0 & .40 & .40 \\ 0 & 0 & 0 & .20 & .40 & .60 & 0 & 0 & .20 \\ 0 & 0 & 0 & 0 & .20 & .40 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .20 & .40 & .60 & .80 & 0 & .20 & .80 \\ 0 & 0 & 0 & .40 & .60 & .60 & 0 & 0 & .20 \\ 0 & 0 & 0 & 0 & .40 & .40 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Table 10
Item Response Submatrices

$$X_{g1} = \begin{array}{c} \\ 4 \\ 1 \\ 9 \\ 2 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad X_{g2} = \begin{array}{c} \\ 5 \\ 3 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Table 11
Reordered Item Response Submatrices

$$X^*_{g1} = \begin{array}{c} \\ 4 \\ 1 \\ 9 \\ 2 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccc} A & E & B & D & C \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad X^*_{g2} = \begin{array}{c} \\ 5 \\ 3 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccc} E & B & C & A & D \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Table 12
Item Dominance Matrices, $A^*_{gh} = X^*_{gh}{}' X_{gh}$

$$A^*_g = \begin{array}{c} \\ A \\ E \\ B \\ D \\ C \end{array} \begin{array}{c} \begin{array}{ccccc} A & E & B & D & C \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Row} \\ \text{Total} \\ 10 \\ 6 \\ 3 \\ 1 \\ 0 \end{array} \qquad A^*_g = \begin{array}{c} \\ E \\ B \\ C \\ A \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} E & B & C & A & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Row} \\ \text{Total} \\ 6 \\ 6 \\ 3 \\ 1 \\ 0 \end{array}$$

Table 13
Order Loading Matrix, C

		Order Dimensions I	II	Row Total
Items	A	10	1	11
	B	3	6	9
	C	0	3	3
	D	1	0	1
	E	6	6	12
Column Total		20	16	36

Figure 3
Graph of Order Loading Matrix for Items A - E

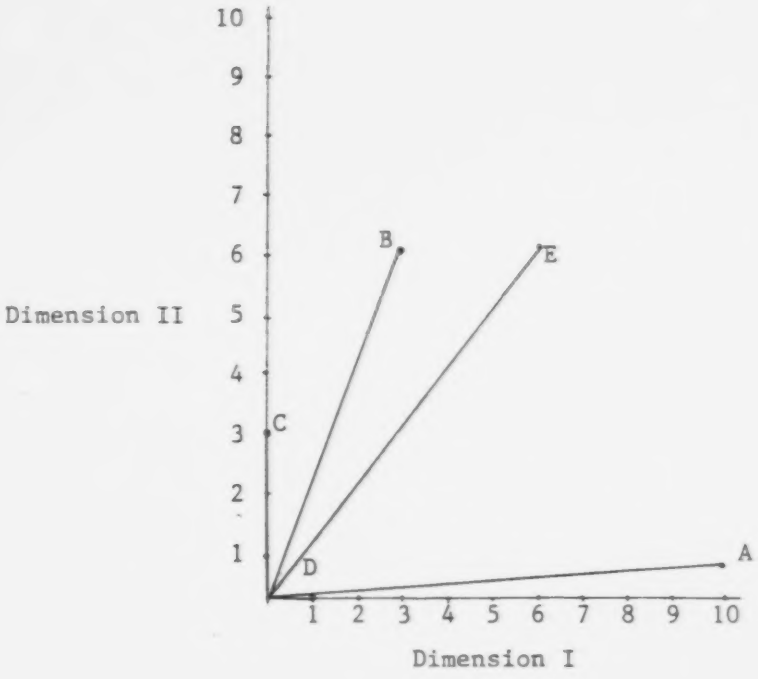

Table 14
Distance Matrix, D

D =	A	B	C	D	E
A	0	7	10	9	5
B	7	0	3	6	3
C	10	3	0	3	6
D	9	6	3	0	6
E	5	3	6	6	0

Table 15
Reordered Item Dominance Matrix, A

A =	A	E	B	C	D
A	0	1	2	4	4
E	2	0	1	4	5
B	2	0	0	3	4
C	1	0	1	0	2
D	0	0	0	1	0

Table 16
Reordered Distance Matrix, D

A =	A	E	B	C	D
A	0	1	2	4	4
E		0	1	4	5
B			0	3	4
C				0	2
D					0

D =	A	E	B	C	D
A	0	5	7	10	9
E		0	3	6	6
B			0	3	6
C				0	3
D					0

Table 17
Item Response Matrix

		Items				
		A	B	C	D	E
Subjects	1	9	7	6	5	4
	2	7	5	4	3	2
	3	4	3	2	1	0
	4	3	4	5	6	8
	5	2	3	4	5	7

Table 18
Matrix P In Partition of Dominance Due To Subjects

P =	1	2	3	4	5
1	0	0	0	5	3
2	10	0	0	10	7
3	21	11	0	17	13
4	10	5	1	0	0
5	13	7	2	5	0

Table 19
Net Dominace Matrix, Q

Q =	1	2	3	4	5
1	0	0	0	0	0
2	10	0	0	5	0
3	21	11	0	16	11
4	5	0	0	0	0
5	10	0	0	5	0

Table 20
Matrix $S_{subject}$

$S_{subject}$ =	1	2	3	4	5
1	0	0	0	0	0
2	100	0	0	25	0
3	441	121	0	256	121
4	25	0	0	0	0
5	100	0	0	25	0

Table 21
Construction of Net Dominance Matrix, H and S_{item} Matrix

$$H = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 3 & 4 & 5 & 4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad S_{item} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 9 & 16 & 25 & 16 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Table 22
Analysis of Variance of Dominance

Source	Dominance	df	SS	MS
Subjects	1214	4	48.56	12.14
Items	74	4	2.96	.74
Residual	1656	16	61.52	3.84
Total	2944	24	113.04	4.71

Table 23
Net Dominance Matrix, Q

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 3 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 3 & 0 \\ 1 & 3 & 2 & 0 & 1 & 4 & 3 & 5 & 2 \\ 0 & 2 & 1 & 0 & 0 & 3 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 3 & 0 \end{bmatrix} \end{array}$$

Table 24
Item Dominance Matrices, A^1 and A^2

$$A^1 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 0 & 2 \\ 2 & 0 & 0 & 0 & 1 \\ 4 & 3 & 0 & 1 & 4 \\ 4 & 4 & 2 & 0 & 5 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad A^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 0 & 3 \\ 2 & 0 & 1 & 0 & 2 \\ 3 & 3 & 0 & 2 & 2 \\ 1 & 1 & 3 & 0 & 1 \\ 2 & 3 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

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